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The 3-D Detection Problem of an Evader Moving in a Fixed Plane

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Abstract—The 3-D detection problem of an evader E by a pursuer P is considered. The evader moves in a plane with constant velocity and along a straight line. A spatial detection zone is attached to the pursuer, whose initial position is outside the evader's plane movement. Pursuer strategies which maximize E's probability of detection are proposed.

1. INTRODUCTION

The following is a general formulation of a detection problem. There is a pursuer, having a detection zone, and an evader. The pursuer's goal is to detect the evader in his detection zone.

The majority of papers [1–4] dedicated to the detection problem consider detection in a plane. In some realistic situations, a 2-D approach to the detection and pursuit-evasion problems is invalid. This is the reason to consider the detection in 3-D [5]. This paper is dedicated to the spatial detection problem.

There are two moving vehicles (players) in 3-D—a pursuer P and an evader E. The evader moves in a plane with a constant velocity V_E along a straight line. The possible directions of his velocity are uniformly distributed on $[0, 2\pi]$. Player P has a spatial detection zone approximated by a quadrangular pyramid. The vertex of this pyramid coincides with P's current position. The base of this pyramid is a rectangle. The altitude r of the pyramid passes through the point of intersection of its base diagonals and coincides with P's velocity vector V_P. The angles included between the apothems of the pyramid's lateral faces are 2δ and 2α. The pursuer's goal is to detect the evader, i.e., to capture the evader in the detection zone.

The pursuer's strategies, when he moves along a straight line with a constant velocity V_P, and controls one (or two) angle(s) which determine the direction of the pursuer's movement, are analyzed in this paper. A strategy when P moves along an arbitrary trajectory, subjected to restrictions on the rate of change of angles which control P's motion, is also suggested. The restriction on P's linear velocity in the last case is $0 \le V_P \le V_P^{\max}$.

2. PROBLEM STATEMENT

We define a fixed rectangular frame $X^0 Y^0 Z^0$. The origin coincides with the pursuer's initial position P^0, and the initial position of the evader E^0 is defined by $(x_E^0, y_E^0, z_E^0 < 0)$ (Figure 1). The evader moves in the plane $z = z_E^0$. The direction of his movement ψ (measured from the x-axis) is a random variable, uniformly distributed in the closed interval $[0, 2\pi]$. We call a plane containing the $P^0 Z^0$ axis and E's initial position as the "vertical plane."

One can define the new fixed rectangular frame $X Y Z$ as follows. The origin is attached to the point P^0, and the abscissa axis goes through the point E^0. The ordinate axis belongs to the vertical plane. The distance between points P^0 and E^0 equals x_0.

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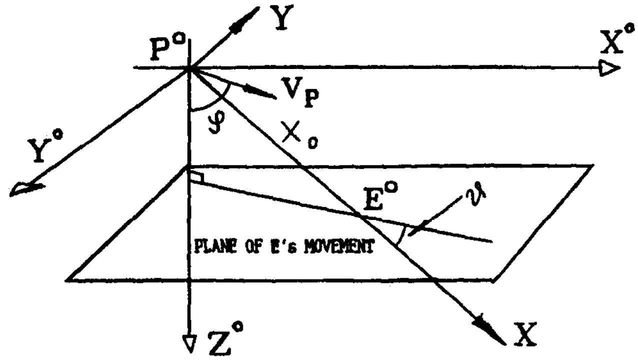

Figure 1.

Let $\Omega(t)$ be the detection zone's (pyramid's) position at the moment t. The pursuer's goal is to detect the evader, i.e., to accomplish the capture of E in the zone $\Omega(t)$ at $t \in [0, \infty]$.

Let's define the payoff functional as:

$$\chi = \begin{cases} 1, & \text{if E will hit the detection zone;} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The criterion is the detection probability, which is the mathematical expectation of the above payoff function, calculated on a measure generated by ψ:

$$G = \mathrm{E}_{\psi}\{\chi\} \longrightarrow \max_{\varphi,\zeta}. \tag{2}$$

Player P maximizes the criterion G. This fact is indicated in (2), where φ and ζ are the angles defining the direction of P's movement in 3-D.

Let ϑ be an angle between the X axis of the frame XYZ and the plane of the evader's movement, $z = z_E^0$. Let's allow the detection zone to intersect with the plane $z = z_E^0$ during the interval $[t_1, t_f]$. The result of this intersection is a trapezium; we name it the "detection trapezium." The position and the shape of this trapezium are dependent upon the parameters of the problem, the initial conditions, and time t.

Let's fix the moment $t \in [t_1, t_f]$. In this connection, E's position in the plane $z = z_E^0$ will be uniformly distributed on a circle (E^0, R), where $R = V_E t$. The joint location of both this circle and the trapezium defines a part of a circle located inside the trapezium. The set of directions of E's movement (angles ψ) corresponding to arcs of this circle, that are located inside the trapezium at any $t \in [t_1, t_f]$, we denote by $S(t)$. The evader's trajectories corresponding to this set lead to detection. Therefore, the detection probability may be defined as

$$G(\varphi) = \left\{ \bigcup_{t \in [t_1, t_f]} S(t) \right\} / 2\pi. \tag{3}$$

To solve the problem, it is necessary to define the set $S(t)$ for each $t \in [t_1, t_f]$ and then to calculate $G(\varphi)$ by means of Formula (3).

At first, a simple variant of Problem (2) is considered, namely, it is suggested that the pursuer moves in the vertical plane, e.g., player P chooses one angle φ between the direction of his movement and the abscissa axis of the frame XYZ (Figure 1). The fact that $\varphi > \vartheta - \frac{\pi}{2}$ is suggested, too. In this case:

$$G = \mathrm{E}_{\psi}\{\chi\} \longrightarrow \max_{\varphi}. \tag{4}$$

Then P's strategy, when he controls two (φ, ζ) angles is considered.

3. SOLUTION

3.1. One Angle Control

Let's call the top rib and the bottom rib of the detection zone those ribs of the pyramid's base which are perpendicular to the vertical plane, as well as those ribs where the top rib is located above the bottom rib in the XYZ frame. Let's define the top and the bottom bases of the detection trapezium: the top base is located above the bottom base in the XYZ frame.

We introduce the following notation (Figure 2):

— a: half of the bottom detection trapezium base length;
— b: half of the top detection trapezium base length;
— c: the distance from the circle (E^0, R) center to the b-base;
— d: the trapezium altitude.

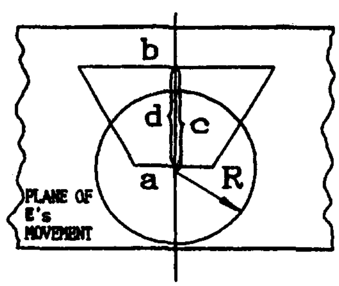

Figure 2.

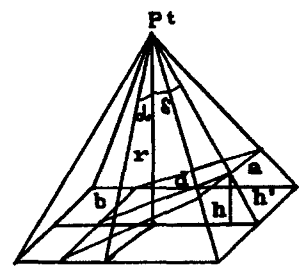

Figure 3.

Let's calculate the time and parameters' dependency of a, b, c, and d. The time segment $[t_1, t_f]$ contains two characteristic values: t^* and t_2. At the moment t^*, the top rib of the detection zone belongs to the plane $z = z_E^0$. At the moment t_2, P's position belongs to this plane. The value of t_1 is defined by means of the condition that the bottom rib of the detection zone belongs to the plane $z = z_E^0$.

Starting from these conditions, one can write down the expressions for t_1, t^*, and t_2:

$$\begin{aligned} t_1 &= \frac{x_0 \cos\delta - r\cos(\varphi - \delta) + r\cot\vartheta \, \sin(\varphi - \delta)}{V_P \cos\delta \, (\cos\varphi - \cot\vartheta \, \sin\varphi)}, \\ t^* &= \frac{x_0 \cos\delta - r\cos(\varphi + \delta) + r\cot\vartheta \, \sin(\varphi + \delta)}{V_P \cos\delta \, (\cos\varphi - \cot\vartheta \, \sin\varphi)}, \\ t_2 &= \frac{x_0}{V_P(\cos\varphi - \cot\vartheta \, \sin\varphi)}. \end{aligned} \tag{5}$$

The value of t_f is defined as $\max(t^*, t_2)$.

It should be pointed out that there are possible situations when $t^* < 0$ and/or $t_1 < 0$. The appearance of these situations means that at the initial moment, the detection zone already interesects with the plane of E's movement. The possibility of such situations has to be taken into account in the solution of the problem.

Let's consider the time interval $[t_1, \min(t^*, t_2)]$. The mutual location of the plane of E's movement and detection zone is shown in Figure 3. According to this figure, one can write the expression for b:

$$b = r\tan\alpha. \tag{6}$$

We convey expressions for a, b, d through h, which is the perpendicular being erected from the bottom trapezium base to the detection zone base. The value h is defined by the formula:

$$h = V_P\,(t - t_1)\,\frac{\cos\delta\,\cos\tilde{\varphi}}{\cos(\tilde{\varphi} - \delta)}, \tag{7}$$

where $\tilde{\varphi} = \frac{\pi}{2} + \varphi - \vartheta$. The values a and d are dependent on h as follows:

$$d = \frac{h}{\sin\tilde{\varphi}}, \qquad \text{and} \tag{8}$$

$$a = b\left(1 - \frac{h}{r}\right). \tag{9}$$

The distance c is given by the formula

$$c = x_0 \frac{\sin(\varphi - \delta)}{\cos(\tilde{\varphi} - \delta)} + V_P\, t \frac{\sin\delta}{\cos(\tilde{\varphi} - \delta)} + d. \tag{10}$$

Then, let $t_2 > t^*$. Note that this case is realized at $\vartheta > \varphi + \delta$ or at $\tilde{\varphi} + \delta < \frac{\pi}{2}$, which is the same. The values a, b, c, d at $t \in [t^*, t_2]$ can be found in the following way. We define the pyramid which differs from the detection zone by altitude only as the sliding detection zone. The top of the sliding detection zone coincides with P's position. The altitude of the sliding detection zone, which we designate by $\tilde{r}$, is defined by the condition that the top rib of the sliding detection zone belongs to the plane of E's movement. So, at $t = t^*$, $\tilde{r} = r$, and at $t > t^*$, $\tilde{r} < r$. The values b, d, and a are defined analogously to (6), (8), and (9); thus,

$$b = \tilde{r}\tan\alpha. \tag{11}$$

A perpendicular h erected from the bottom trapezium base to the sliding detection zone base is

$$h = 2\tilde{r}\,\frac{\sin\delta\sin\tilde{\varphi}}{\cos(\tilde{\varphi} - \delta)}.$$

The trapezium altitude and the half of its bottom base length a are expressed through h as

$$d = \frac{h}{\sin\tilde{\varphi}}, \tag{12}$$

$$a = b\left(1 - \frac{h}{\tilde{r}}\right). \tag{13}$$

The distance c depends on $\tilde{r}$ as follows:

$$c = \frac{\tilde{r}\tan\delta + x_0\sin\varphi}{\cos\tilde{\varphi}}. \tag{14}$$

In Formulas (11)–(14), the sliding detection zone altitude $\tilde{r}$ equals

$$\tilde{r} = r - V_P\,(t - t^*)\,\frac{\cos\tilde{\varphi}\cos\delta}{\cos(\tilde{\varphi} + \delta)}. \tag{15}$$

In the case $t_2 < t^*$, for the time interval $[t_2, t^*]$, the value c is defined as

$$c = x_0 \frac{\sin\varphi}{\cos\tilde{\varphi}} - V_P\, t \frac{\sin\delta}{\cos(\tilde{\varphi} + \delta)} + d. \tag{16}$$

For b, d, and a, Formulas (6), (8), and (9) are correct. In these formulas,

$$h = r + V_P\,(t - t_2)\,\frac{\cos\delta\cos\tilde{\varphi}}{\cos(\tilde{\varphi} + \delta)}. \tag{17}$$

In the case when the value t_1 calculated by Formula (5) is negative and t^* is positive, Formulas (6) and (8)–(10) are correct in the interval $[t_1, \min(t^*, t_2)]$. In this case, one must regard the value h_0 as a segment that is parallel to the pursuer's velocity vector, and that connects the plane of the evader's movement and bottom detection zone rib at the initial moment, namely:

$$h = \{V_P\,(t - t_1) + h_0\}\,\frac{\cos\delta\cos\tilde{\varphi}}{\cos(\tilde{\varphi} - \delta)}. \tag{18}$$

In addition, the time t_1 is assumed to be equal to 0. The distance h_0 is

$$h_0 = \left\{\frac{r}{\cos\delta} - x_0\,\frac{\sin\vartheta}{\cos(\tilde{\varphi} - \delta)}\right\}\frac{\cos(\tilde{\varphi} - \delta)}{\cos\tilde{\varphi}}. \tag{19}$$

If the values t_1 and t^* are negative, Formulas (11)–(15) are correct in the interval $[t^*, t_2]$ with regard to the change of r to r_0; r_0 is the altitude of the sliding detection zone at the initial moment. In this regard, we assume that $t^* = 0$, in accordance with the definition

$$r_0 = x_0\,\frac{\sin\vartheta\cos\delta}{\sin(\frac{\pi}{2} + \tilde{\varphi} + \delta)}. \tag{20}$$

We now propose the following algorithm for a, b, c, and d calculation.

I. Values t_1, t^*, t_2 are calculated by Formulas (5). If $t_1 < 0$ and $t^* > 0$, then $t_1 = 0$, and h_0 is calculated by (19); else $h_0 = 0$. If $t_1 < 0$ and $t^* < 0$, then $t_1 = 0$, $t^* = 0$, and r_0 is calculated by (20); else $r_0 = r$.

II. For a fixed time $t \in [t_1, t_f]$, where $t_f = \max(t^*, t_2)$, the values a, b, c, and d are calculated according to the following conditions.
   - If $t < t^*$ and $t < t_2$, Formulas (6)–(10) are used and h is defined by (18).
   - Else, if $t_2 > t^*$, Formulas (11)–(15) are applied and r has to be changed to r_0 in these formulas.

   In the remaining cases, Formulas (16) and (6), (8), (9), with regard to (17), are used.

Detection probability calculation

The sum of the lengths of the arcs located inside the trapezium is proportional to the sum of the central angles corresponding to these arcs. All these angles are counted from the line going through E^0, and which is perpendicular to the trapezium bases. The angles ψ_a, ψ_b, ψ_l, and ψ_h are formed by straight lines drawn from E^0 through points of the circle, the bottom trapezium base, the top trapezium base, and the trapezium lateral side intersection, respectively. Angle ψ_b is counted counterclockwise and the other angles, clockwise. Also, $\psi_h > \psi_l$. The values of these angles are defined by the formulas

$$\begin{aligned}
\psi_b &= \arccos\frac{c}{R},\\
\psi_a &= \arccos\frac{d-c}{R},\\
\psi_h &= \arccos\frac{\lambda\,\mu - \sqrt{\mu^2 - \lambda^2 + 1}}{\mu^2 + 1},\\
\psi_l &= \arccos\frac{\lambda\,\mu + \sqrt{\mu^2 - \lambda^2 + 1}}{\mu^2 + 1},
\end{aligned}$$

where

$$\mu = \frac{b-a}{d}, \qquad \lambda = \frac{a + (d-c)\mu}{R}.$$

The set $S(t)$ is the unification of segments

$$S(t) = [-\psi_b, \psi_b] \cup [-\psi_a, \psi_a] \cup [\psi_l, \psi_h] \cup [-\psi_h, -\psi_l].$$

Problem (4) is solved by numerical methods, namely, for each φ, one value $G(\varphi)$ is found; the optimal control angle φ^* corresponds to the maximum value $G(\varphi)$.

In order to find the probability $G(\varphi)$ by Formula (3), it is possible to find the set $S(t)$ for each moment t. Beforehand, it is necessary to consider all possible variants of the mutual location of both the circle and the trapezium, as well as the conditions of their realization. All together there are twelve variants in the considered case. The calculation of the set $S(t)$ for each variant differs from the others.

For practical realization, the above procedure for the calculation of $G(\varphi)$ must contain an algorithm of unification of an arbitrary amount of segments. That complicates the probability calculation problem. Therefore, a simpler probability calculation procedure is used in the computation. For each $t \in [t_1, t_f]$, the cycle $\psi \in [0, \pi]$ is made. For each ψ, the following test is made. It is checked whether the point of circle (E^0, R) (which corresponds to the given ψ) is inside the trapezium, a test of its belonging to the existing set S is made, and if the point does not belong to the set S, then this set is extended.

The given procedure is sufficiently simple in its realization and it does not require the calculation of the values ψ_a, ψ_b, ψ_h, ψ_l and of the conditions for the realization of different variants.

The result of the calculation, i.e., the dependence of $G(\varphi)$ is given in Figure 4 (curve I). In the computations, the following parameters were used: $V_P = 3$, $V_E = 2$, $r = 60$, $X_0 = 90$, $\alpha = 14°$, $\delta = 10°$, and $\vartheta = 48.2°$.

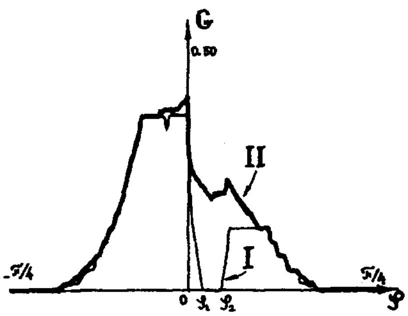

Figure 4.

3.2. Two Angles Control

We first define the angles giving the direction of P's movement. One of them, φ, was already defined. Central to the second angle definition are the following concepts.

Let's assume that a second angle ζ is the angle between P's velocity vector and the vertical plane. Then, while varying ζ, the shape and parameters of the detection trapezium will be changed, too. This considerably complicates the probability calculation; e.g., it would be desirable to choose as the second angle one for which its variation does not change the shape and parameters of the trapezium. Starting from this, we define the second angle, ζ.

We fix angle φ. Let's consider a cone, the top of which coincides with P^0. The altitude is perpendicular to the plane of the evader's movement, and an element of the cone is a straight line containing P's velocity vector, being located on the vertical plane. We call this cone the cone of the pursuer's movement. Then, we will let the pursuer choose the direction of his motion so that his velocity vector coincides with an element of the cone. In this case, the shape and parameters of the trapezium do not change with the variation of ζ.

Let F be the intersection point of the altitude of the cone and the plane of the evader's motion. Point K is the intersection point of the cone element containing the velocity vector V_P, and the plane of E s movement. Through points F and K, we draw a straight line. We define angle ζ as the angle between this straight line and a straight line that is the intersection of the vertical plane and the plane of E's movement. This angle defines the direction of the pursuer's movement

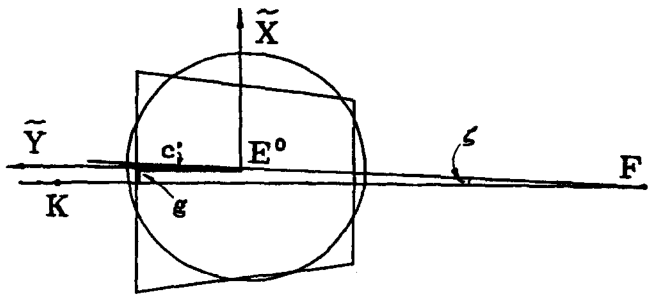

Figure 5.

on the cone. At $\zeta = 0$, the pursuer moves in the vertical plane (cf. Section 3.1); the shape and parameters of the detection trapezium do not change with the ζ variation (Figure 5).

Construction of the probability calculation algorithm in the case when the two angles (φ, ζ) are controlled

In constrast to the case $\zeta = 0$ (Section 3.1), the problem is asymmetric with respect to the straight line (F, E^0) at $\zeta \neq 0$. On account of this, additional (in comparison with the case $\zeta = 0$) variants of mutual location of both the circle (E^0, R) and the detection trapezium appear. Furthermore, the angles ψ_a^1, ψ_b^1, ψ_h^1, and ψ_l^1, located on one side of straight line (E^0, F) are not equal to the angles ψ_a^2, ψ_b^2, ψ_h^2, and ψ_l^2, located on the other side.

In the calculation of the probability, it is possible to use the algorithm that distinguishes different variants of mutual location of the detection trapezium and the circle (E^0, R). One can calculate this probability using expressions for ψ_a, ψ_b, ψ_h, and ψ_l as follows:

$$\psi_{h,l}^1 = \arccos \frac{\lambda\,\mu\cos\zeta \pm \sqrt{\mu^2 - \lambda^2 + 1 + 2\mu\sin\zeta}\,(1 + \mu\sin\zeta)}{\mu^2 + 1 + 2\mu\sin\zeta},$$
$$\psi_{h,l}^2 = \arccos \frac{\lambda\,\mu\cos\zeta \pm \sqrt{\mu^2 - \lambda^2 + 1 - 2\mu\sin\zeta}\,(1 - \mu\sin\zeta)}{\mu^2 + 1 + 2\mu\sin\zeta},$$

where

$$\mu = \left\{\frac{b-a}{d} - \tan\zeta\right\}\cos\zeta,$$
$$\lambda = \frac{\cos\zeta}{R}\left\{a + \frac{b-a}{d}\left[\ell\left(\cos\zeta - 1 - \sin\zeta\,\frac{d}{b-a}\right) + d - c\right]\right\}.$$

The symbol ℓ, designating the distance $|\mathrm{F}, E^0|$, is defined as

$$\ell = x_0\cos\vartheta. \tag{21}$$

On account of the awkwardness of the formulas and the appearance of additional variants, the probability calculation based on this algorithm is very inconvenient. Thus, a more simple algorithm is suggested.

We define a new frame $\tilde{X}\tilde{Y}$: the ordinate axis constitutes the angle ζ with the line (F, E^0). The origin coincides with E^0. The mutual location of the detection trapezium and the circle is characterized by the circle radius $R = V_E t$, the trapezium parameters (parameters a, b, d are given by formulas from Section 3.1), and also by the distance between the circle's center and the top trapezium base, c', and by the distance g between the $\tilde{Y}$-axis and a straight line connecting the midpoints of the trapezium bases. The values c' and g are defined by the formulas:

$$g = \ell\sin\zeta, \tag{22}$$
$$c' = \ell(1 - \cos\zeta) + c. \tag{23}$$

The equations for the lateral sides of detection trapezium in XY coordinates are

$$\begin{array}{ll} \text{right side} & \tilde{y} = k_0\,\tilde{x} + k_1, \\ \text{left side} & \tilde{y} = -k_0\,\tilde{x} + k_2, \end{array} \tag{24}$$

with coefficients:

$$k_0 = \frac{d}{b-a}, \quad k_1 = c' - (b-g)k_0, \quad k_2 = c' - (b+g)k_0.$$

Let ψ be the angle constituted by E's velocity vector and the $\tilde{Y}$ axis. This angle is counted clockwise, and at $\psi = 0$, E's velocity vector is anti-co-linear to the $\tilde{Y}$-axis. Then, the following equation system is correct for the angle $\hat{\psi}$, corresponding to the evader's trajectory, where the latter is detected:

$$\begin{aligned} &y' = -R\cos(\hat{\psi} + \zeta), \\ &x' = -R\sin(\hat{\psi} + \zeta), \\ &y' < c', \quad y' > c' - d, \quad y' > k_0 x' + k_1, \quad y' > k_0 x' + k_2. \end{aligned} \tag{25}$$

The procedure for estimating the set S, and hence for the calculation of the detection probability, is analogous to the procedure of calculation of the probability in the case where the pursuer controls one angle only. The difference is that the angle ψ must vary from 0 to 2π, due to the problem's asymmetry.

4. RESULTS AND DISCUSSION

Let's consider P's control strategy of the two angles (φ, ζ), where P chooses the direction of his motion at the initial moment, and where φ and ζ are constant, i.e., P's trajectory is a straight line. In Figure 4, the dependencies $G(\varphi)$ at $\zeta = 0$ (curve I) and at $\zeta = \text{constant} \neq 0$ (curve II) are plotted. In the last case, the angle ζ for every φ was calculated, and so its value, which corresponds to the maximum probability value for a given φ, was chosen.

Note that the value of φ^* in the case under consideration equals $-0.5°$. At $x_0 = 100$ (the other initial conditions remained unchanged), $\varphi^* = -3°$.

If the angle φ cannot be chosen exactly, and it is chosen with an error $\Delta\varphi$, then in order to provide a detection probability close to the optimal value, φ has to belong to the segment $[\varphi^* - \Delta\varphi, \varphi^*]$. This follows from the dependence of $G(\varphi)$.

Segment $[\varphi_1, \varphi_2]$ (Figure 4, with $\zeta = 0$) corresponds to the case where the detection trapezium is located inside the circle (E^0, R) during the time interval $[t_1, t_f]$. The introduction of the additional control angle ζ, as follows from the diagram, allows to improve the detection quality for some angles φ, for example, for $\varphi \in [\varphi_1, \varphi_2]$. However, for the optimal φ^*, the pursuer's gain, or an increase of the detection probability, is not practically improved by introducing the second control angle ζ.

On account of this, a detection strategy using a wider set of pursuer's trajectories, namely, $\varphi = \varphi(t)$, $\zeta = \zeta(t)$, $V_P = V_P(t)$, is suggested. The pursuer chooses his motion parameters to satisfy the following condition: at first, the detection trapezium sweeps over a semicircle located on one side of a straight line (F, E^0) (this corresponds, for example, to $\zeta > 0$). In this connection, the distance from the point F to the top trapezium base changes from $(\ell - R(t))$ to $(\ell + R(t))$. Then, the trapezium sweeps over the semicircle located on the other side $(\zeta < 0)$. The distance from point F to the top trapezium base changes, in this case, from $(\ell + R(t))$ to $(\ell - R(t))$. The pursuer's velocity has to be changed to provide the existence of the detection trapezium.

It should be pointed out that a consideration of this strategy is meaningful if restrictions on the rate of change of angles φ, ζ and restricitons on the linear velocity $0 \leq V_P \leq V_P^{\max}$ are introduced.

The dependencies of the pursuer's velocity and of the angles φ and ζ upon time that are obtained as a result of the above strategy calculation are depicted in Figure 6. The probability calculated for this strategy equals 1. In the probability calculation, the restrictions to the rate of change of angles were in the range $0 \le |\varphi| \le 3$, $0 \le |\zeta| \le 7$. It was also assumed that $|V_P| \le 3$ and $V_P^{\max} = 3$.

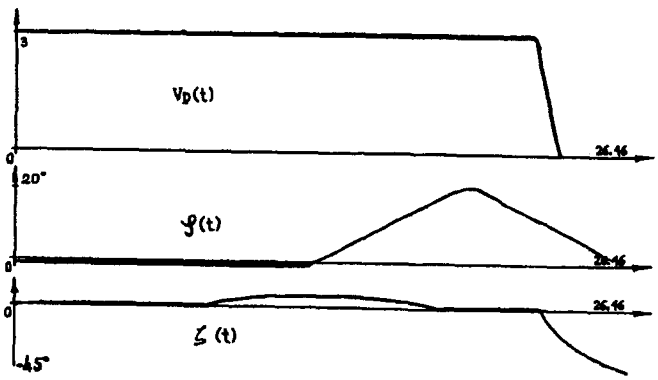

Figure 6.

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